

q -Painlevé systems arising from q -KP hierarchy

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Abstract

A system of q -Painlevé type equations with multi-time variables t_1, \dots, t_M is obtained as a similarity reduction of the N -reduced q -KP hierarchy. This system has affine Weyl group symmetry of type $A_{M-1}^{(1)} \times A_{N-1}^{(1)}$. Its rational solutions are constructed in terms of q -Schur functions.

1 Introduction

Fix an integer $N \geq 3$. The following system of q -difference equations has been introduced in [2]:

$$\begin{aligned} \overline{a_i} &= a_i, \quad \overline{x_i} = \frac{x_{i-1}}{a_i} \frac{\varphi_{i-1}(x)}{\varphi_{i+1}(x)}, \\ \varphi_i(x) &= 1 + x_{i-1} + x_{i-2}x_{i-1} + \cdots + x_{i-N+1} \cdots x_{i-1}, \end{aligned} \tag{1}$$

for $i = 0, 1, \dots, N-1$. Here a_i and x_i ($i = 0, 1, \dots, N-1$) are constant parameters and dependent variables satisfying $a_{i+N} = a_i$ and $x_{i+N} = x_i$, respectively. The symbol \overline{f} denotes the discrete time evolution of a variable f . We put $a_0 a_1 \cdots a_{N-1} = q^{-N}$ and $x_0 x_1 \cdots x_{N-1} = t$; t plays the role of independent variable such that $\overline{t} = q^N t$.

The system (1) has the Bäcklund transformations $s_0, s_1, \dots, s_{N-1}, \pi$ defined

as follows:

$$\begin{aligned}
\pi(a_i) &= a_{i+1}, & \pi(x_i) &= x_{i+1}, \\
s_i(a_{i-1}) &= a_{i-1}a_i, & s_i(x_{i-1}) &= \frac{x_{i-1}}{a_i} \frac{1+a_ix_i}{1+x_i}, \\
s_i(a_i) &= \frac{1}{a_i}, & s_i(x_i) &= a_ix_i, \\
s_i(a_{i+1}) &= a_ia_{i+1}, & s_i(x_{i+1}) &= x_{i+1} \frac{1+x_i}{1+a_ix_i}, \\
s_i(a_j) &= a_j, & s_i(x_j) &= x_j \quad (j \neq i, i \pm 1).
\end{aligned} \tag{2}$$

These transformations generate the affine Weyl group of type $A_{N-1}^{(1)}$.

The case of $N = 3$ corresponds to the q -Painlevé IV equation (q - P_{IV})[1] through the change of variables $a_i \rightarrow a_i^2$, $x_i \rightarrow f_i/a_i$ and $q \rightarrow q^{-1/3}$. The case of $N = 4$ is considered as a q -difference analog of Painlevé V equation (q - P_V), whose rational solutions are studied in [3].

The aim of this paper is to introduce a hierarchy of these equations with multi-time variables t_1, \dots, t_M . This hierarchy, formulated as a similarity reduction of an N -reduced q -KP hierarchy, has the affine Weyl group symmetry of type $A_{M-1}^{(1)} \times A_{N-1}^{(1)}$. We also construct certain rational solutions expressed in terms of q -Schur functions by using this formulation.

2 q -KP hierarchy

We first introduce a q -difference analog of the KP hierarchy (q -KP hierarchy).

Let t_i and T_i ($i = 1, 2, \dots, M$) be the time variables and their q -shift operators:

$$T_i(t_i) = qt_i, \quad T_i(t_j) = t_j \quad (i \neq j). \tag{3}$$

We also use the notation such as $T_i T_j \cdots T_k = T_{i,j,\dots,k}$. For $i = 1, 2, \dots, M$, we define the $\mathbb{Z} \times \mathbb{Z}$ matrices

$$B_i = U_i + t_i \Lambda, \tag{4}$$

where

$$U_i = \text{diag}(u_{i,j})_{j \in \mathbb{Z}}, \quad \Lambda = (\delta_{i+1,j})_{i,j \in \mathbb{Z}}, \tag{5}$$

and consider the linear q -difference equations

$$T_k \Psi = B_k \Psi, \quad (k = 1, \dots, M) \quad \Psi = (\psi_i)_{i \in \mathbb{Z}}. \tag{6}$$

We define the q -KP hierarchy as a system of nonlinear q -difference equations for the unknown variables $u_{i,j} = u_{i,j}(t_1, \dots, t_M)$, through the compatibility condition of (6),

$$T_k(B_i)B_k = T_i(B_k)B_i \quad (i, k = 1, \dots, M). \tag{7}$$

The evolution equation for $u_{i,j}$ with respect to t_k is expressed explicitly as

$$T_k(u_{i,j}) = u_{i,j} \frac{t_i u_{k,j+1} - t_k u_{i,j+1}}{t_i u_{k,j} - t_k u_{i,j}} \quad (i \neq k). \tag{8}$$

Introduce the τ -functions τ_j ($j \in \mathbb{Z}$) by

$$u_{i,j} = \frac{T_i(\tau_j)\tau_{j-1}}{T_i(\tau_{j-1})\tau_j}. \quad (9)$$

These τ functions are defined up to normalization constants. By substituting (9) into (8), we have

Theorem 2.1 *Under an appropriate normalization, the q -KP hierarchy (8) is described by the Hirota-Miwa bilinear q -difference equations*

$$t_i T_i(\tau_{k-1})T_j(\tau_k) - t_j T_j(\tau_{k-1})T_i(\tau_k) = (t_i - t_j)T_{ij}(\tau_{k-1})\tau_k. \quad (10)$$

A class of rational solutions of the q -KP hierarchy is described in terms of the q -Schur functions as follows. Define $h_k(t)$ ($k = 0, 1, 2, \dots$) by

$$\sum_{k=0}^{\infty} h_k(t) z^k = \prod_{i=1}^M \frac{1}{(t_i z; q)_{\infty}}, \quad (11)$$

or

$$h_k(t) = \sum_{\nu_1 + \dots + \nu_M = k} \frac{t_1^{\nu_1} \dots t_M^{\nu_M}}{(q; q)_{\nu_1} \dots (q; q)_{\nu_M}}. \quad (12)$$

We set $h_k(t) = 0$ ($k < 0$). For a sequence of integers $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_l)$, the q -Schur function $S_{\lambda}(t)$ is defined as

$$S_{\lambda}(t) = \det \left[h_{\lambda_i - i + j}(t) \right]_{1 \leq i, j \leq l}. \quad (13)$$

For $\lambda = (\lambda_1, \dots, \lambda_l)$ and $k \in \mathbb{Z}$, we put $(k, \lambda) = (k, \lambda_1, \dots, \lambda_l)$. For two sequences of integers $\lambda = (\lambda_1, \dots, \lambda_l)$ and $\mu = (\mu_1, \dots, \mu_m)$, we say that $\lambda \equiv \mu$ if $S_{\lambda} = \pm S_{\mu}$. For instance we have $(\dots, a, b, \dots) \equiv (\dots, b-1, a+1, \dots)$ and $(\dots, a, b, 0) \equiv (\dots, a, b)$.

The following Proposition asserts that the q -Schur functions solve the Hirota-Miwa equation (10).

Proposition 2.2 *For any k and λ , we have*

$$t_i T_i(S_{\lambda})T_j(S_{(k, \lambda)}) - t_j T_j(S_{\lambda})T_i(S_{(k, \lambda)}) = (t_i - t_j)T_{ij}(S_{\lambda})S_{(k, \lambda)}. \quad (14)$$

Proof. Define a semi-infinite matrix Φ with indices $i, j = 1, 2, \dots$ as

$$\Phi = \sum_{l=0}^{\infty} h_l(t) \Lambda^l, \quad \Lambda_{i,j} = \delta_{i+1,j}. \quad (15)$$

For any partition $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_l)$, $\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_l \geq 0$, the q -Schur function $S_{\lambda}(t)$ is the minor determinant of the matrix Φ with rows $(1, 2, \dots, l)$ and columns $(\lambda_l + 1, \lambda_{l-1} + 2, \dots, \lambda_1 + l)$. Note that the matrix Φ solves the q -difference system

$$T_n(\Phi) = (1 - t_n \Lambda) \Phi \quad (n = 1, 2, \dots, M). \quad (16)$$

Then the relation (14) reduces to the determinant identity (Plücker relation) in Lemma 2.3 below for $X = \Phi\Sigma$ defined with an appropriate permutation matrix Σ . \square

Lemma 2.3 *For any matrix X , we have*

$$(a - b)\det_n(V_a V_b X)\det_{n+1}(V_c X) + (abc \text{ cyclic}) = 0. \quad (17)$$

Where $V_a = 1 - a\Lambda$ and $\det_n(X)$ is the $n \times n$ minor determinant of the matrix X with rows and columns $(1, 2, \dots, n)$.

We note that the q -Schur functions also satisfy the following bilinear q -difference equation.

Proposition 2.4 *For any partition λ and $k < l \in \mathbb{Z}$, we have*

$$t_i S_{(l-1, k, \lambda)} T_i(S_\lambda) + S_{(k, \lambda)} T_i(S_{(l, \lambda)}) - S_{(l, \lambda)} T_i(S_{(k, \lambda)}) = 0. \quad (18)$$

Proof. This is also a consequence of the following determinant identity. \square

Lemma 2.5 *For any matrix X , we have*

$$a\xi^{(I, k, l)}(X)\xi^I(V_a X) + \xi^{(I, k)}(X)\xi^{(I, l)}(V_a X) - \xi^{(I, l)}(X)\xi^{(I, k)}(V_a X) = 0, \quad (19)$$

where $\xi^J(X)$ is a minor determinant of X with rows $(1, 2, \dots, p)$ and columns $J = (j_1, \dots, j_p)$.

3 Reduction to q -Painlevé equations

We consider two kinds of reduction conditions for the q -KP hierarchy, namely, the N -reduction and the similarity reduction.

We first impose N -periodicity,

$$u_{i, j+N} = u_{i, j}, \quad \tau_{j+N} = c_j \tau_j, \quad (20)$$

for some constants c_j (N -reduction). Correspondingly the Lax formalism is rewritten in terms of $N \times N$ matrices with a spectral parameter z as follows:

$$T_k \Psi = B_k \Psi, \quad B_k = U_k + t_k \Lambda, \quad (21)$$

where $\Psi = (\psi_i)_{1 \leq i \leq N}$, $U_i = \text{diag}(u_{i,1}, \dots, u_{i,N})$ and $\Lambda = \sum_{i=1}^{N-1} E_{i, i+1} + z E_{N,1}$.

We next impose the similarity condition. Consider the action of the Euler operator $T_{1, \dots, M}$:

$$T_{1, \dots, M} \Psi = B_{1, \dots, M} \Psi, \quad (22)$$

where

$$B_{1, \dots, M} = T_{2, \dots, M}(B_1) T_{3, \dots, M}(B_2) \cdots T_M(B_{M-1}) B_M. \quad (23)$$

Setting

$$A = K^{-1}B_{1,\dots,M}, \quad K = \text{diag}(q^{N-1}, q^{N-2}, \dots, 1), \quad (24)$$

we require that Ψ should obey the similarity condition,

$$T_{q^N, z}\Psi = A\Psi, \quad (25)$$

where $T_{q^N, z}$ is the q -shift operator with respect to z such that $T_{q^N, z}(z) = q^N z$. The compatibility condition

$$T_{q^N, z}(B_k)A = T_k(A)B_k \quad (26)$$

of (25) and (21) implies in fact the homogeneity of $u_{i,j}$:

$$T_{1,\dots,M}(u_{i,j}) = u_{i,j}. \quad (27)$$

Consistently, we may assume

$$T_{1,\dots,M}(\tau_j) = \gamma_j \tau_j, \quad (28)$$

for some constants γ_j .

The N -reduced q -KP hierarchy with similarity condition gives a system of q -difference equations of the form

$$T_k(u_{i,j}) = F_{k,i,j}(u), \quad (k, i = 1, \dots, M, j = 1, \dots, N), \quad (29)$$

which we call q -Painlevé system of type (M, N) . In (29) $F_{k,i,j}(u)$ are in general complicated rational functions of the variables $u_{i,j}$ and t_i ($i = 1, \dots, M, j = 1, \dots, N$). Their explicit form will be given in the next section.

4 The Weyl group realization

The time evolution and symmetry of the q -Painlevé system can be described in the framework of a birational action of the affine Weyl group of type $A_{M-1}^{(1)} \times A_{N-1}^{(1)}$. To this end, we first discuss a realization of the affine Weyl group as a group of automorphisms of a field of rational functions, apart from the context of the q -KP hierarchy.

We introduce a new set of variables $x_{i,j}$ ($i = 1, \dots, M, j = 1, \dots, N$) and define the action of the affine Weyl group on the field of rational functions $K = \mathbb{C}(x)$ in MN variables $x_{i,j}$. We extend the indices i, j of $x_{i,j}$ to $i, j \in \mathbb{Z}$ by the condition $x_{i+M,j} = qx_{i,j}$ and $x_{i,j+N} = px_{i,j}$ with some fixed constants q and p .

Define algebra automorphisms π, ρ, r_i and s_j ($i \in \mathbb{Z}/M\mathbb{Z}, j \in \mathbb{Z}/N\mathbb{Z}$) on

the field K as follows:

$$\begin{aligned}
\pi(x_{i,j}) &= x_{i+1,j}, & \rho(x_{i,j}) &= x_{i,j+1}, \\
r_i(x_{i,j}) &= px_{i+1,j} \frac{P_{i,j-1}}{P_{i,j}}, & r_i(x_{i+1,j}) &= p^{-1}x_{i,j} \frac{P_{i,j}}{P_{i,j-1}}, \\
r_k(x_{i,j}) &= x_{i,j}, \quad (k \neq i, i+1), \\
s_j(x_{i,j}) &= qx_{i,j+1} \frac{Q_{i-1,j}}{Q_{i,j}}, & s_j(x_{i,j+1}) &= q^{-1}x_{i,j} \frac{Q_{i,j}}{Q_{i-1,j}}, \\
s_k(x_{i,j}) &= x_{i,j}, \quad (k \neq j, j+1),
\end{aligned} \tag{30}$$

where

$$\begin{aligned}
P_{i,j} &= \sum_{a=1}^N \left(\prod_{k=1}^{a-1} x_{i,j+k} \prod_{k=a+1}^N x_{i+1,j+k} \right), \\
Q_{i,j} &= \sum_{a=1}^M \left(\prod_{k=1}^{a-1} x_{i+k,j} \prod_{k=a+1}^M x_{i+k,j+1} \right).
\end{aligned} \tag{31}$$

Theorem 4.1 *The automorphisms $\langle \pi, r_0, r_1, \dots, r_{M-1} \rangle$ and $\langle \rho, s_0, s_1, \dots, s_{N-1} \rangle$ generate the extended affine Weyl group $\widetilde{W}(A_{M-1}^{(1)})$ and $\widetilde{W}(A_{N-1}^{(1)})$, respectively. Moreover these two actions $\widetilde{W}(A_{M-1}^{(1)})$ and $\widetilde{W}(A_{N-1}^{(1)})$ mutually commute.*

This theorem is proved for the special case of $p = q = 1$ in [2] (see also [4]). We omit the proof of Theorem 4.1 since essentially the same argument applies to the case of general p and q .

Let Γ_i ($i = 1, \dots, M$) be the translations in $\widetilde{W}(A_{M-1}^{(1)})$ defined by

$$\Gamma_i = r_{i-1}r_{i-2} \cdots r_1 \pi r_{M-1} \cdots r_{i+1} r_i. \tag{32}$$

Then Γ_i define a commuting family of discrete flows, for which $\widetilde{W}(A_{N-1}^{(1)})$ acts as Bäcklund transformations. By a computation similar to that given in [5], explicit formulas for the actions of Γ_i are obtained as follows:

$$\Gamma_i(x_{k,j}) = \begin{cases} p^{M-1} q x_{k,j} \frac{P_j^{k+1, \dots, M+i}}{P_j^{k+1, \dots, M+i}} & (k = i), \\ p^{-1} x_{k,j} \frac{P_j^{k, M+i}}{P_j^{k-1, M+i}} & (k = M + i - 1), \\ p^{-1} x_{k,j} \frac{P_j^{k+1, \dots, M+i} P_j^{k, \dots, M+i}}{P_j^{k+1, \dots, M+i} P_j^{k-1, \dots, M+i}} & (\text{otherwise}). \end{cases} \tag{33}$$

Here

$$P_j^{i, \dots, i+r} = \sum_a \left(\prod_{k=1}^{a_1-1} x_{i,j+k} \right) \left(\prod_{k=a_1+1}^{a_2-1} x_{i+1,j+k} \right) \cdots \left(\prod_{k=a_r+1}^{rN} x_{i+r,j+k} \right), \tag{34}$$

and the sum \sum_a is taken over integers a_i such that $0 \leq a_1 < a_2 < \dots < a_r \leq rN$ and $1 \leq a_i - a_{i-1} \leq N$. As is shown in [2], the system for $(M, N) = (2, N)$ for $p = 1$ is equivalent to (1).

We now give an explicit correspondence between the above family of discrete flows and the q -Painlevé systems described in the previous section.

Theorem 4.2 *With the notation of the q -Painlevé system of type (M, N) as in Section 3, define the variables $x_{i,j}$ by*

$$x_{i,j} = \frac{1}{t_i} T_{i+1, \dots, M}(u_{i,j}) \quad (i = 1, \dots, M, j = 1, \dots, N). \quad (35)$$

Then, in terms of the x variables, the time evolutions of the q -Painlevé system coincide with the commuting flows defined by Γ_k^{-1} ($k = 1, \dots, M$) under the birational action of $\widetilde{W}(A_M^{(1)}) = \langle r_0, \dots, r_{M-1}, \pi \rangle$ with $p = 1$. Namely, we have

$$T_k(x_{i,j}) = \Gamma_k^{-1}(x_{i,j}) \quad (i = 1, \dots, M, j = 1, \dots, N). \quad (36)$$

In order to prove Theorem 4.2, the following lemma which was originally given in [2] plays a crucial role.

Lemma 4.3 *(Lemma 3.1 of [2]) For given MN variables $x_{i,j}$ and MN unknowns $x'_{i,j}$ ($i = 1, \dots, M, j = 1, \dots, N$), we denote $X_i = \text{diag}(x_{i,1}, \dots, x_{i,N})$ and $X'_i = \text{diag}(x'_{i,1}, \dots, x'_{i,N})$. Then the system of algebraic equations*

$$(X_1 + \Lambda) \cdots (X_M + \Lambda) = (X'_1 + \Lambda) \cdots (X'_M + \Lambda), \quad (37)$$

has $M!$ solutions. Each of the solution corresponds to a permutation $\sigma \in S_M$ and characterized by additional conditions

$$x'_{i,1} x'_{i,2} \cdots x'_{i,N} = x_{\sigma(i),1} x_{\sigma(i),2} \cdots x_{\sigma(i),N} \quad (i = 1, \dots, M). \quad (38)$$

Moreover, if σ is given as a product $\sigma = \sigma_{i_1} \cdots \sigma_{i_k}$, where $\sigma_i = (i, i+1)$, then the corresponding solution is explicitly expressed as

$$x'_{i,j} = r_{i_1} \cdots r_{i_k}(x_{i,j}), \quad (39)$$

by means of the birational Weyl group action of $W(A_{M-1}^{(1)}) = \langle r_1, \dots, r_{M-1} \rangle$ with $p = 1$ (30).

Proof of Theorem 4.2. We first remark that, for each $i = 1, \dots, M$, the product $u_{i,1} \cdots u_{i,N}$ is invariant with respect to the q -shift operators T_k ($k = 1, \dots, M$). Hence, we have

$$x_{i,1} \cdots x_{i,N} = t_i^{-N} T_{i+1, \dots, M}(u_{i,1} \cdots u_{i,N}) = t_i^{-N} u_{i,1} \cdots u_{i,N} \quad (40)$$

and

$$T_k(x_{i,1} \cdots x_{i,N}) = q^{-N\delta_{i,k}} x_{i,1} \cdots x_{i,N}. \quad (41)$$

We now prove that, for each $k = 1, \dots, M$, the action of T_k^{-1} on the x variables coincides with that of $\Gamma_k = r_{k-1} \cdots r_1 \pi r_{M-1} \cdots r_k$ (with $p = 1$). Since $T_{1,\dots,M} = T_k T_M T_{M-1} \cdots \widehat{T_k} \cdots T_1$, the matrix $B_{1,\dots,M}$ is expressed alternatively as

$$B_{1,\dots,M} = T_{2,\dots,M}(B_1) T_{3,\dots,M}(B_2) \cdots T_{k,\dots,M}(B_{k-1}) \cdot T_k [T_{k+2,\dots,M}(B_{k+1}) \cdots T_M(B_{M-1}) B_M] B_k. \quad (42)$$

This implies

$$(X_1 + \Lambda) \cdots (X_M + \Lambda) = (X_1 + \Lambda) \cdots (X_{k-1} + \Lambda) \cdot T_k [(X_{k+1} + \Lambda) \cdots (X_M + \Lambda)] (t_k^{-1} U_k + \Lambda). \quad (43)$$

In the right-hand side, the products of diagonal entries of individual factors are arranged as

$$\begin{aligned} x_{i,1} \cdots x_{i,N} & \quad (i = 1, \dots, k-1), \\ T_k(x_{i,1} \cdots x_{i,N}) &= x_{i,1} \cdots x_{i,N} \quad (i = k+1, \dots, N), \\ t_k^{-N} u_{k,1} \cdots u_{k,N} &= x_{k,1} \cdots x_{k,N}. \end{aligned} \quad (44)$$

according to the cyclic permutation $(k, k+1, \dots, M)$. Hence by Lemma 4.1, we obtain

$$\begin{aligned} x_{i,j} &= r_k \cdots r_{M-1}(x_{i,j}) \quad (i = 1, \dots, k-1), \\ T_k(x_{i+1,j}) &= r_k \cdots r_{M-1}(x_{i,j}) \quad (i = k, \dots, M-1), \\ t_k^{-1} u_{k,j} &= r_k \cdots r_{M-1}(x_{M,j}), \end{aligned} \quad (45)$$

for all $j = 1, \dots, N$. Similarly, from $T_{1,\dots,M} = T_M \cdots \widehat{T_k} \cdots T_1 T_k$, we obtain

$$B_{1,\dots,M} = T_k^{-1} [T_{1,\dots,M}(B_k) T_{2,\dots,M}(B_1) T_{3,\dots,M}(B_2) \cdots T_{k,\dots,M}(B_{k-1})] \cdot T_{k+2,\dots,M}(B_{k+1}) \cdots T_M(B_{M-1}) B_M, \quad (46)$$

hence

$$(X_1 + \Lambda) \cdots (X_M + \Lambda) = (t_k^{-1} T_k^{-1}(U_k) + \Lambda) T_k^{-1} [(X_1 + \Lambda) \cdots (X_{k-1} + \Lambda)] \cdot (X_{k+1} + \Lambda) \cdots (X_M + \Lambda). \quad (47)$$

This time the factorization on the right-hand side corresponds to the cyclic permutation $(k, k-1, \dots, 1)$; hence we have

$$\begin{aligned} t_k^{-1} T_k^{-1}(u_{k,j}) &= r_{k-1} \cdots r_1(x_{1,j}), \\ T_k^{-1}(x_{i-1,j}) &= r_{k-1} \cdots r_1(x_{i,j}) \quad (i = 2, \dots, k), \\ x_{i,j} &= r_{k-1} \cdots r_1(x_{i,j}) \quad (i = k+1, \dots, M) \end{aligned} \quad (48)$$

for all $j = 1, \dots, N$. By using (45) and (48), we can determine the action of T_k^{-1} as follows:

$$\begin{aligned} T_k^{-1} r_k \cdots r_{M-1}(x_{i,j}) &= T_k^{-1}(x_{i,j}) = r_{k-1} \cdots r_1(x_{i+1,j}) \quad (i = 1, \dots, k-1), \\ T_k^{-1} r_k \cdots r_{M-1}(x_{i,j}) &= x_{i+1,j} = r_{k-1} \cdots r_1(x_{i+1,j}) \quad (i = k, \dots, M-1), \\ T_k^{-1} r_k \cdots x_{M,j} &= q t_k^{-1} T_k^{-1}(u_{k,j}) = r_{k-1} \cdots r_1(q x_{1,j}), \end{aligned} \quad (49)$$

for all $j = 1, \dots, N$. Namely we have

$$T_k^{-1} r_k \cdots r_{M-1}(x_{i,j}) = r_{k-1} \cdots r_1 \pi(x_{i,j}) \quad (i = 1, \dots, M, j = 1, \dots, N). \quad (50)$$

This means that the action of T_k^{-1} on the x variables coincides with that of $\Gamma_k = r_{k-1} \cdots r_1 \pi r_{M-1} \cdots r_k$. \square

We now construct the rational solutions for the q -Painlevé system.

We say that a sequence of partitions $\lambda(i)$ ($i \in \mathbb{Z}/N\mathbb{Z}$) is an N -reduced chain if $\lambda(i) \equiv (k_i, \lambda(i-1))$ for some $k_i \in \mathbb{Z}$. For instance $\lambda(0) = (1, 1)$, $\lambda(1) = (2, 1, 1)$ and $\lambda(2) = (2, 2, 1, 1)$ is an example of 3-reduced chain, where $k_1 = 2$, $k_2 = 2$, $k_3 = k_0 = -4$.

For any N -reduced chain $\lambda(i) = (\lambda_1(i), \lambda_2(i), \dots)$ ($i \in \mathbb{Z}/N\mathbb{Z}$) given, we put $\tau_i = S_{\lambda(i)}(t_1, \dots, t_M)$. Then from Proposition 2.2, the τ functions τ_i ($i \in \mathbb{Z}/N\mathbb{Z}$) give a similarity solution of the N -reduced q -KP hierarchy such that $T_{1,\dots,M}(\tau_i) = \gamma_i \tau_i$ with $\gamma_i = |\lambda(i)| = \lambda_1(i) + \lambda_2(i) + \dots$.

Corollary 4.4 *For any N -reduced chain of τ functions τ_i defined as above, we put*

$$x_{i,j} = \frac{1}{t_i} \frac{T_{i,\dots,M}(\tau_j) T_{i+1,\dots,M}(\tau_{j-1})}{T_{i+1,\dots,M}(\tau_j) T_{i,\dots,M}(\tau_{j-1})} \quad (i = 1, \dots, M, j = 1, \dots, N). \quad (51)$$

Then $x_{i,j}$ solve the q -Painlevé system of type (M, N)

$$x_{i,j}(t_1, \dots, q^{-1}t_k, \dots, t_M) = \Gamma_k(x_{i,j}) \quad (k = 1, \dots, M). \quad (52)$$

We finally remark on the Toda type bilinear q -difference equations satisfied by q -Schur functions. For simplicity, we consider the case of $M = 2$, namely the case with two time variables $t = (t_1, t_2)$. For a q -Schur function $\tau = S_\lambda(t)$ let us put $\tau_{k,\dots,l} = S_{(l,\dots,k,\lambda)}(t)$. It follows from Proposition 2.4 that

$$\begin{aligned} t_1 \tau_{k,l} T_1(\tau) + \tau_k T_1(\tau_l) - \tau_l T_1(\tau_k) &= 0, \\ t_2 \tau_{k,l} T_2(\tau) + \tau_k T_2(\tau_l) - \tau_l T_2(\tau_k) &= 0. \end{aligned} \quad (53)$$

Applying T_1 on the second equation, and using the similarity condition, we have

$$t_2 T_1(\tau_{k,l}) \tau + T_1(\tau_k) q^l \tau_l - T_1(\tau_l) q^k \tau_k = 0. \quad (54)$$

Combining (54) and the first equation of (53), we obtain

$$q^k t_1 T_1(\tau) \tau_{kl} + t_2 T_1(\tau_{kl}) \tau = (q^k - q^l) T_1(\tau_k) \tau_l. \quad (55)$$

Define monic polynomial $Q_\lambda(x)$ as $Q_\lambda(x) = c_\lambda^{-1} S_\lambda(x, 1)$. Here the normalization constants are given, in terms of the hook-length [6] $h_{i,j} = \lambda_i - j + \lambda'_j - i + 1$ of the partition λ , as

$$c_\lambda = q^{\sum_i (i-1)\lambda_i} \prod_{(i,j) \in \lambda} (1 - q^{h_{i,j}})^{-1}. \quad (56)$$

Then (55) yields

$$Q_{(k,\lambda)}(qx)Q_{(l,\lambda)}(x) = Q_{(l-1,k,\lambda)}(qx)Q_{\lambda}(x) + xq^k Q_{\lambda}(qx)Q_{(l-1,k,\lambda)}(x). \quad (57)$$

In the 3-reduced case, (57) coincides with the Toda equations for the q -Okamoto polynomials([2] Theorem 2.7). At the same time, from the determinant formula (13), we obtain the Jacobi-Trudi type formula for the q -Okamoto polynomials; in this particular case, the entries are essentially the continuous q -Hermite polynomials.

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